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farlane. He points out that in the algebra of the plane there is no need of specifying the axis of rotation for a circular angle, such as e^{ib}. It is different in space. When angles on a sphere are considered, the axis must be inserted, and also $\pi/2$ for the indefinite i.[1] If by α^a we mean a radians round the axis α, α^1 means 1 radian round α. Here $\log \alpha^1 = \alpha^{\pi/2}$ and $\log \alpha^a = a \log \alpha^1 = a\alpha^{\pi/2}$. Whence $\alpha^a = e^{a \log_e \alpha^1} = e^{a\alpha^{\pi/2}}$. Now Hamilton adopted the principle which, in Macfarlane's notation is $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta + \sin\alpha\beta[\alpha\beta]^{\pi/2}$, where $[\alpha\beta]$ signifies the unit which is normal to α and β. Macfarlane, on the other hand, adopts the principle $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta - \sin\alpha\beta\,[\alpha\beta]^{\pi/2}$. By this modification Macfarlane is able to establish for his space-algebra the simple theorem which Hamilton could not derive from his assumption, $\log(e^{a\alpha^{\pi/2}} \cdot e^{b\beta^{\pi/2}}) = a\alpha^{\pi/2} + b\beta^{\pi/2}$.

A SIMPLE FORMULA FOR THE ANGLE BETWEEN TWO PLANES.

By EDWARD V. HUNTINGTON, Harvard University.

Let us consider the familiar problem of finding the area of a surface by the methods of the integral calculus.

Let S be any closed portion of a surface whose equation is $z = f(x, y)$, and let ΔS be an "element" of the surface at any point P. Then, as is well known,

$$\Delta S = \frac{\Delta S'}{\cos\gamma},$$

where $\Delta S'$ is the projection of ΔS on the xy-plane, and γ is the angle between the xy-plane and the tangent plane to the surface at P. The area of S is therefore given by

$$S = \int\int \frac{dS'}{\cos\gamma},$$

where the integration is to be extended over the whole of the plane area S' which is the projection of S on the xy-plane.

The problem of finding the area S thus reduces to a simple problem in double integration, *provided the value of* γ *at every point of the surface is known.*

In the ordinary treatment of the subject in the text-books, the derivation of the formula for γ requires a considerable knowledge of analytical geometry of three dimensions, including the equation of a plane, the equations of a straight line normal to a plane, the method of finding the direction cosines of a line when its equations are given, the equation of a tangent plane, etc.

The object of this note is *to show how the angle* γ *may be found directly*, by very elementary methods. The formula obtained, in the easily-remembered form here given, and the methods of derivation are so simple that it is hardly possible that they are new; and yet I do not find them in any of the current text-books.

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[1] A. Macfarlane, "A vector-analysis as generalized algebra," *International Congress of Mathematicians*, Cambridge, August, 1912; "Account of researches in the algebra of physics," *Journal of the Washington Acad. of Sci.*, Vol. II, Nos. 14, 15, 16, 1912.

THE FORMULA FOR THE ANGLE γ.

Let P be any point of the surface, and let PA and PB (see either figure) be the lines in which the tangent plane at P intersects the vertical planes drawn through P parallel to the planes xz and yz respectively.

The angles φ_x and φ_y which PA and PB make with the horizontal can be readily determined from the fact that tan φ_x is the partial derivative of z with respect to x, and tan φ_y is the partial derivative of z with respect to y; thus,

$$\tan \varphi_x = f_x'(x, y) = D_x z, \quad \tan \varphi_y = f_y'(x, y) = D_y z.$$

We may therefore suppose that tan φ_x and tan φ_y are known for every point P of the surface.

Then if γ is the angle which the tangent plane at P makes with the base plane, the value of tan γ may be at once computed by the easily remembered formula

$$\mathbf{\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y,}$$

whence, of course, $\sec \gamma = \sqrt{1 + \tan^2 \gamma} = \sqrt{1 + (D_x z)^2 + (D_y z)^2}$.

The proof of the formula may be obtained by either of two methods, as follows.

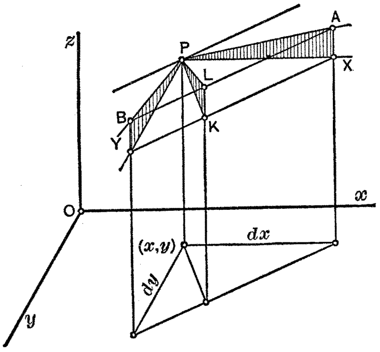

FIG. 1.

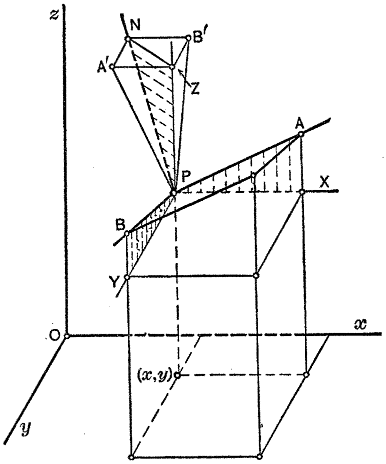

FIG. 2.

First Proof.—Referring to figure 1, we may suppose that the values of dx and dy have been so chosen that the perpendiculars AX and BY dropped from A and B on the horizontal plane through P shall be equal, say $AX = BY = h$. Then

$$\tan \varphi_x = \frac{h}{dx} \quad \text{and} \quad \tan \varphi_y = \frac{h}{dy}.$$

Now since $AX = BY$, the lines AB and XY will be parallel to the line of intersection of the tangent plane with the horizontal plane. If, therefore, we draw PK perpendicular to XY and PL perpendicular to AB, then PK and PL will be perpendicular to the edge of the dihedral angle γ, so that the angle KPL will be the correct measure of that dihedral angle. Hence

$$\tan \gamma = \frac{h}{PK}.$$

But, by considering the right triangle PXY, we find at once that

$$\frac{1}{PK^2} = \frac{1}{dx^2} + \frac{1}{dy^2}.$$

Therefore

$$\tan^2 \gamma = \frac{h^2}{PK^2} = \frac{h^2}{dx^2} + \frac{h^2}{dy^2} = \tan^2 \varphi_x + \tan^2 \varphi_y,$$

which was to be proved.

Second Proof.—Through the given point P, in figure 2, draw PX, PY, PZ parallel to the coördinate axes, and let PN be the normal to the surface at P, so that $\angle ZPN = \gamma$.

In the plane PXZ draw PA' perpendicular to PA, so that $\angle ZPA' = \angle XPA = \varphi_x$, and in the plane PYZ draw PB' perpendicular to PB, so that $\angle ZPB' = \angle YPB = \varphi_y$; and let N, A', B', be the points of the lines PN, PA', PB' which lie in a horizontal plane through Z. Then

$$\tan \gamma = \frac{ZN}{PZ}, \qquad \tan \varphi_x = \frac{ZA'}{PZ}, \qquad \tan \varphi_y = \frac{ZB'}{PZ}.$$

If now we can show that the figure $ZA'NB'$ is a rectangle, we shall have

$$ZN^2 = ZA'^2 + ZB'^2,$$

and hence $\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y$, as desired.

We prove first that NA' is parallel to ZB'.

From the figure, PN is perpendicular to PA, PA' is perpendicular to PA, and PY is perpendicular to PA. Hence PA' lies in the plane PNY. But this plane cuts the horizontal plane through Z in a line NA' which is parallel to PY and hence parallel to ZB'.

Similarly we can prove that NB' is parallel to ZA'.

Hence the figure $ZA'NB'$ is at least a parallelogram. But the angle between ZA' and ZB' is known to be a right angle. Hence the figure is a rectangle, and the proof is complete.

It will be noticed that the distinction between the two methods of proof consists mainly in the fact that the required angle γ is regarded as the *dihedral angle* between the planes in the first proof, and as the *angle between the normals* to those planes in the second proof. These angles are of course equal.

A modification of the first proof may be obtained by considering *two* planes perpendicular to the edge of the dihedral, one containing AX and the other containing AY, instead of the single plane containing P.